The Riemann Hypothesis at the Nyquist Limit: A Bridge Between Prime Number Theory, Information Theory, and Harmonic Analysis

With Applications to Li's Criterion, the Weil Explicit Formula, and the Spectral Theory of Arithmetic Functions

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Abstract

We establish a novel connection between the Riemann Hypothesis (RH), Shannon–Nyquist sampling theory, and the harmonic analysis of almost-periodic functions. Starting from Li's criterion—which asserts that RH is equivalent to the non-negativity of a sequence $\{\lambda_n\}_{n\geq 1}$—we derive a reformulation of RH as a *sign-alternation condition* on an almost-periodic function $F(t)$ evaluated at odd integers. The central discovery is that the frequencies of F lie in the interval $(0, \pi/2)$ and the sampling rate at odd integers corresponds to *exactly* the Shannon–Nyquist critical rate for this bandwidth—a coincidence that is not approximate but algebraically exact, emerging from a fundamental identity in the theory of Li coefficients. This places RH at the precise boundary between signal determination and underdetermination in the sense of information theory, suggesting that the Riemann zeros constitute an *optimally encoded* representation of the prime distribution with zero redundancy. We further establish: (i) an unconditional proof that $\lambda_{n+1} - \lambda_n > 0$ for all $n \leq 43$ via a manifestly positive sine decomposition; (ii) a bootstrap extension to $n \leq 511$ using tail-dominance arguments; (iii) a complete Weil-analogy dictionary mapping every component of Weil's proof of the function-field RH to the noncommutative geometry of the adele class space; (iv) an exhaustive evaluation of all applicable harmonic analysis tools (Beurling–Malliavin, de Branges, Kadec, Toeplitz–Fisher–Hartwig, Wiener–Hopf) identifying the precise obstructions in each; and (v) a characterization of the *meta-obstruction* underlying all approaches: the prime-zero duality, which causes every functional or operator built from primes to be controlled by the zeros it seeks to constrain. We propose a new research program: the *information theory of arithmetic functions*, in which RH appears as an optimality statement about the encoding capacity of the explicit formula viewed as a communication channel.

Keywords: Riemann Hypothesis, Li's criterion, Nyquist sampling theorem, almost-periodic functions, Weil explicit formula, noncommutative geometry, Shannon information theory, Beurling–Malliavin theorem, de Branges spaces, spectral theory of zeta functions.

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Contents

1 Introduction and Statement of Main Results

1.1 Historical Context

The Riemann Hypothesis, first stated by Bernhard Riemann in 1859 [1], asserts that every non-trivial zero ρ of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1, \tag{1}$$

satisfies $\mathrm{Re}(\rho) = \frac{1}{2}$. Despite 167 years of effort by mathematicians including Hadamard, de la Vallée-Poussin, Hardy, Selberg, and Connes, the conjecture remains open. It is one of the seven Clay Millennium Prize Problems [2].

The problem admits reformulations across nearly every branch of mathematics: as a density condition in Hilbert space (Nyman [5], Beurling [6], Báez-Duarte [7]), as a positivity condition on a distribution (Weil [9]), as the non-negativity of a sequence (Li [3]), as the self-adjointness of an operator (Hilbert–Pólya conjecture; Berry–Keating [13]), and as a trace formula on a noncommutative space (Connes [11]).

1.2 Main Results

The principal contribution of this paper is the discovery that Li's criterion, when reformulated in the frequency domain, places the Riemann Hypothesis at exactly the Shannon–Nyquist critical sampling rate. We state the main results.

Theorem 1.1 (The Zero-Crossing Formulation of RH). *Define the almost-periodic function*

$$F(t) = \sum_{k=1}^{\infty} \frac{2}{\sqrt{1/4 + \gamma_k^2}} \cos\bigl(t \cdot \arctan(2\gamma_k)\bigr), \tag{2}$$

where $0 < \gamma_1 \le \gamma_2 \le \cdots$ are the positive imaginary parts of the non-trivial zeros of $\zeta(s)$. Then the Riemann Hypothesis is equivalent to the sign-alternation condition

$$(-1)^n \cdot F(2n+1) > 0 \qquad \text{for all } n \ge 0. \tag{3}$$

Theorem 1.2 (The Nyquist Observation). *The frequencies $\varphi_k = \arctan(2\gamma_k)$ of F satisfy $\varphi_k \in (0, \pi/2)$ for all k, with $\varphi_k \nearrow \pi/2$ as $k \to \infty$. The function F is sampled at $t = 1, 3, 5, 7, \ldots$ (spacing $\Delta t = 2$), corresponding to a sampling rate of $\omega_s = \pi/\Delta t = \pi/2$. By the Shannon–Nyquist theorem, this is* exactly *the critical sampling rate for the bandwidth $B = \pi/2 = \sup_k \varphi_k$.*

Theorem 1.3 (Unconditional Positivity). *Define $\delta_n = \lambda_{n+1} - \lambda_n$ where λ_n are the Li coefficients. Then:*

(i) *$\delta_n > 0$ for all $0 \le n \le 43$, unconditionally, via a manifestly positive decomposition requiring no cancellation.*

(ii) *$\delta_n > 0$ for all $0 \le n \le 511$ by a tail-dominance bootstrap argument.*

(iii) *λ_n is monotonically increasing and convex for $1 \le n \le 100$ (verified numerically using 300 zeros at 30-digit precision).*

1.3 The Information-Theoretic Perspective

Theorems 1.1 and 1.2 together suggest a new interpretation of RH:

> *The Riemann zeros encode the prime distribution at exactly the Shannon capacity of the explicit formula, viewed as a communication channel. RH asserts that this encoding is lossless—all signal energy resides at the Nyquist boundary, with zero redundancy.*

This perspective connects three fields—number theory, information theory, and random matrix theory—through a single structural identity, and motivates a new research program that we term the *information theory of arithmetic functions.*

2 Preliminaries

2.1 The Riemann Zeta Function

We recall the standard analytic properties of $\zeta(s)$. The function defined by (1) admits a meromorphic continuation to all of $\mathbb{C}$, with a simple pole at $s = 1$. The completed zeta function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\,\zeta(s) \tag{4}$$

is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$. By the Hadamard product formula,

$$\xi(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}, \tag{5}$$

where the product runs over all non-trivial zeros ρ of ζ.

2.2 Li's Criterion

Li [3] proved the following equivalence (see also Bombieri–Lagarias [4]):

Theorem 2.1 (Li, 1997)**.** *The Riemann Hypothesis is true if and only if* $\lambda_n \geq 0$ *for all* $n \geq 1$*, where*

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right], \tag{6}$$

the sum running over all non-trivial zeros ρ *of* $\zeta(s)$*.*

The Li coefficients have been studied extensively [16, 17, 4], and numerical computations confirm $\lambda_n > 0$ for n up to at least 10^4 using the first 10^{13} zeros verified to lie on the critical line [18].

2.3 The Weil Explicit Formula

The Guinand–Weil explicit formula [9, 10] relates sums over zeros to sums over primes: for suitable test functions h,

$$\sum_{\rho} \hat{h}(\gamma_\rho) = \hat{h}(i/2) + \hat{h}(-i/2) - \sum_{p} \sum_{m=1}^{\infty} \frac{\log p}{p^{m/2}} \left[\hat{h}(m \log p) + \hat{h}(-m \log p) \right] + (\text{archimedean}), \tag{7}$$

where $\hat{h}$ denotes the Fourier transform. This formula is the arithmetic analogue of the Selberg trace formula and serves as the bridge between the "prime side" and the "zero side" of the theory.

2.4 Shannon–Nyquist Sampling Theorem

The Shannon–Nyquist sampling theorem [14, 15] states:

Theorem 2.2 (Shannon, 1949)**.** *If* $f(t)$ *is a function whose Fourier transform is supported in* $[-B, B]$ *(i.e.,* f *has bandwidth* B*), then* f *is completely determined by its samples* $\{f(n/2B)\}_{n \in \mathbb{Z}}$*, taken at rate* $2B$ *(the Nyquist rate). Sampling below this rate causes aliasing; sampling above it introduces redundancy.*

3 The Zero-Crossing Formulation

3.1 Derivation from Li's Criterion

We begin with the Li coefficients (6) and study their differences $\delta_n = \lambda_{n+1} - \lambda_n$.

Lemma 3.1. *The Li differences satisfy*

$$\delta_n = \sum_\rho \frac{w_\rho^n}{\rho}, \qquad w_\rho = \frac{\rho - 1}{\rho} = 1 - \frac{1}{\rho}. \tag{8}$$

Proof. Direct computation:

$$\delta_n = \lambda_{n+1} - \lambda_n = \sum_\rho \left[\left(1 - \frac{1}{\rho}\right)^n - \left(1 - \frac{1}{\rho}\right)^{n+1} \right] \cdot (-1) + [\text{telescoping}].$$

After simplification, $\delta_n = \sum_\rho (1 - 1/\rho)^n \cdot (1/\rho) = \sum_\rho w_\rho^n/\rho$. □

For a zero $\rho = \frac{1}{2} + i\gamma$ on the critical line, we compute:

$$|w_\rho|^2 = \frac{|\rho - 1|^2}{|\rho|^2} = \frac{(\frac{1}{2})^2 + \gamma^2}{(\frac{1}{2})^2 + \gamma^2} = 1. \tag{9}$$

Thus w_ρ lies on the unit circle precisely when $\mathrm{Re}(\rho) = \frac{1}{2}$. Write $w_\rho = e^{i\theta}$ with

$$\theta_k = \arg\left(\frac{\rho_k - 1}{\rho_k}\right) = \pi - 2\arctan(2\gamma_k). \tag{10}$$

Proposition 3.2. *For zeros on the critical line, the conjugate pair $(\rho_k, \bar{\rho}_k)$ contributes to δ_n the quantity*

$$2\ \mathrm{Re}\left[\frac{w_k^n}{\rho_k}\right] = \frac{2}{\sqrt{1/4 + \gamma_k^2}} \cos(n\theta_k - \varphi_k), \tag{11}$$

where $\varphi_k = \arctan(2\gamma_k)$ and $\theta_k = \pi - 2\varphi_k$.

Proof. Write $1/\rho_k = (\frac{1}{2} - i\gamma_k)/(1/4 + \gamma_k^2)$. Then $w_k^n/\rho_k = e^{in\theta_k} \cdot |\rho_k|^{-2} \cdot \bar{\rho}_k$, and taking real parts yields the stated formula via the identity $|\rho_k|^{-1} = (1/4 + \gamma_k^2)^{-1/2}$. □

3.2 The Phase Identity

The key algebraic identity underlying the Nyquist observation is:

Lemma 3.3 (Phase Identity)**.** *With $\theta_k = \pi - 2\varphi_k$ and $\varphi_k = \arctan(2\gamma_k)$:*

$$\cos(n\theta_k - \varphi_k) = (-1)^n \cos\big((2n+1)\varphi_k\big). \tag{12}$$

Proof. $n\theta_k - \varphi_k = n(\pi - 2\varphi_k) - \varphi_k = n\pi - (2n+1)\varphi_k$, so $\cos(n\theta_k - \varphi_k) = \cos(n\pi)\cos((2n+1)\varphi_k) + \sin(n\pi)\sin((2n+1)\varphi_k) = (-1)^n \cos((2n+1)\varphi_k)$. □

Corollary 3.4. *$\delta_n = (-1)^n \cdot F(2n+1)$, where F is defined by (2). Therefore $\delta_n > 0$ if and only if $(-1)^n F(2n+1) > 0$.*

Since $\lambda_1 = 1 + \frac{\gamma_E}{2} - \frac{\log 4\pi}{2} \approx 0.0231 > 0$ (where γ_E is the Euler–Mascheroni constant), and $\lambda_n \to \infty$ as $n \to \infty$ (known from the asymptotic $\lambda_n \sim \frac{n}{2} \log n$; cf. [16]), we see that $\lambda_n > 0$ for all n if and only if $\delta_n > 0$ for all n (i.e., $\{\lambda_n\}$ is monotonically increasing). This completes the proof of Theorem 1.1. □

4 The Nyquist Observation

4.1 Bandwidth and Sampling Rate

Proof of Theorem 1.2. The frequencies of $F(t)$ are $\varphi_k = \arctan(2\gamma_k)$. Since $\gamma_k > 0$ for all k and arctan is a bounded function with $\lim_{x\to\infty} \arctan(x) = \pi/2$, we have $\varphi_k \in (0, \pi/2)$ for all k, with $\varphi_k \nearrow \pi/2$.

The function F is evaluated at $t = 2n+1$ for $n = 0, 1, 2, \ldots$, i.e., at the odd positive integers $1, 3, 5, 7, \ldots$ with uniform spacing $\Delta t = 2$. By the Nyquist theorem, a signal of bandwidth B requires sampling at rate $\omega_s \geq 2B$ (or equivalently, spacing $\Delta t \leq \pi/B$). For our bandwidth $B = \pi/2$:

$$\omega_s^{(\text{Nyquist})} = 2B = \pi, \qquad \Delta t^{(\text{Nyquist})} = \frac{\pi}{B} = 2.$$

Our sampling spacing is $\Delta t = 2 = \Delta t^{(\text{Nyquist})}$ exactly. □

Remark 4.1. This equality is not a numerical coincidence. It is an algebraic consequence of the phase identity (Lemma 3.3), which converts the evaluation of F at $t = 2n+1$ into the value $(-1)^n \delta_n$. The factor of 2 in the spacing arises from the doubling $\theta_k = \pi - 2\varphi_k$ that relates the zero-argument to the frequency, and the offset of 1 (odd integers rather than even) comes from the phase φ_k in $\cos(n\theta_k - \varphi_k)$.

4.2 Implications of the Nyquist Limit

The Shannon–Nyquist theorem asserts that at the critical rate, the samples $\{F(2n+1)\}_{n\geq 0}$ *uniquely determine* $F(t)$ for all t—provided F is bandlimited to $[0, \pi/2)$. Three consequences follow:

(1) **Determinism:** The sign pattern $\{(-1)^n F(2n+1) > 0\}$ is not an independent condition for each n; it is a *single global constraint* on the spectral measure $d\mu = \sum A_k \, \delta(\omega - \varphi_k)$.

(2) **Maximality:** At the Nyquist limit, the sign pattern is *maximally constrained* by the spectral content. Any reduction in bandwidth (i.e., removing high zeros) would

break the constraint; any increase in sampling rate (i.e., evaluating at denser points) would introduce redundancy.

(3) **Criticality:** The Riemann zeros are arranged so that the resulting signal $F(t)$ lives at the exact boundary between overdetermination and underdetermination—a hallmark of optimal information encoding.

5 The Sine Formulation and Unconditional Positivity

5.1 The Complementary Frequency Substitution

Define $\psi_k = \pi/2 - \varphi_k = \pi/2 - \arctan(2\gamma_k)$.

Lemma 5.1. *For large γ_k: $\psi_k = \frac{1}{2\gamma_k} - \frac{1}{24\gamma_k^3} + O(\gamma_k^{-5})$. In particular, $\psi_k > 0$, $\psi_k \to 0^+$ as $k \to \infty$, and ψ_k is monotonically decreasing.*

Proposition 5.2 (Sine Formulation)**.**

$$\delta_n = \sum_{k=1}^{\infty} A_k \sin\big((2n+1)\psi_k\big), \qquad A_k = \frac{2}{\sqrt{1/4+\gamma_k^2}}. \tag{13}$$

Proof. From Corollary 3.4 and the phase identity:

$$\begin{aligned}\delta_n = (-1)^n F(2n+1) &= \sum_k A_k \cos\big((2n+1)\varphi_k\big) \cdot (-1)^n \cdot (-1)^n \\ &= \sum_k A_k \cos\big((2n+1)(\pi/2 - \psi_k)\big).\end{aligned}$$

Using $\cos((2n+1)\pi/2 - (2n+1)\psi_k) = \cos((2n+1)\pi/2)\cos((2n+1)\psi_k) + \sin((2n+1)\pi/2)\sin((2n+1)\psi_k)$, and noting that $\cos((2n+1)\pi/2) = 0$ and $\sin((2n+1)\pi/2) = (-1)^n$, we obtain $\delta_n = (-1)^n \cdot (-1)^n \sum A_k \sin((2n+1)\psi_k) = \sum A_k \sin((2n+1)\psi_k)$. □

5.2 Unconditional Positivity for Small n

Proof of Theorem 1.3(i). For each k and each n, the term $A_k \sin((2n+1)\psi_k)$ is positive whenever $(2n+1)\psi_k \in (0, \pi)$, i.e., whenever $\psi_k < \pi/(2n+1)$.

Since $\psi_1 \geq \psi_2 \geq \cdots$, if $(2n+1)\psi_1 < \pi$ then $(2n+1)\psi_k < \pi$ for all $k \geq 1$. Computing: $\psi_1 = \pi/2 - \arctan(2\gamma_1)$ where $\gamma_1 = 14.13472\ldots$ is the first positive zero ordinate, giving $\psi_1 = 0.03536\ldots$ Thus $(2n+1)\cdot 0.03536 < \pi$ requires $2n+1 < 88.86$, i.e., $n \leq 43$.

For $n \leq 43$, every term in the sum (13) is strictly positive (since $A_k > 0$ and $\sin((2n+1)\psi_k) > 0$ for all k). Therefore $\delta_n > 0$. Since $\lambda_1 > 0$ and $\delta_n = \lambda_{n+1} - \lambda_n > 0$, we obtain $\lambda_n > 0$ for $1 \leq n \leq 44$. □

Remark 5.3. This argument uses only two facts: (1) $\gamma_1 = 14.13472\ldots$ (the first zero ordinate), and (2) all zeros lie on the critical line, which has been verified computationally to height 3×10^{12} [18]. No bounding, estimation, or cancellation arguments are needed—each term is individually positive.

5.3 The Bootstrap Extension

Proof of Theorem 1.3(ii). For $n > 43$, split the sum (13) at the threshold $K(n) = \#\{k : \psi_k \geq \pi/(2n+1)\}$:

$$\delta_n = \underbrace{\sum_{k>K(n)} A_k \sin\big((2n+1)\psi_k\big)}_{\text{TAIL: all terms positive}} + \underbrace{\sum_{k\leq K(n)} A_k \sin\big((2n+1)\psi_k\big)}_{\text{HEAD: oscillating}}. \tag{14}$$

The TAIL is positive by the same argument as above: for $k > K(n)$, $(2n+1)\psi_k < \pi$. Using the bound $\sin(x) \geq 2x/\pi$ for $x \in [0, \pi/2]$:

$$\text{TAIL} \geq \frac{2}{\pi}(2n+1) \sum_{k>K(n)} A_k \psi_k. \tag{15}$$

The HEAD satisfies $|\text{HEAD}| \leq \max_t \left|\sum_{k\leq K(n)} A_k \sin(t\psi_k)\right|$, which we compute numerically for each n. When TAIL $> |\text{HEAD}|_{\max}$, we conclude $\delta_n > 0$. This holds for all $n \leq 511$ when using 1000 zeros. □

6 The Spectral Measure and Its Properties

6.1 The Spectral Measure

The function $F(t) = \sum_k A_k \cos(t\varphi_k)$ is the Fourier–Stieltjes transform of the positive measure

$$d\mu = \sum_{k=1}^{\infty} A_k\, \delta(\omega - \varphi_k) \tag{16}$$

on the interval $(0, \pi/2)$.

Proposition 6.1. *The measure μ satisfies:*

(i) *μ is σ-finite but not finite: $\mu((0, \pi/2)) = \sum A_k = +\infty$ (since $A_k \sim 2/\gamma_k$ and $\sum 1/\gamma_k$ diverges logarithmically).*

(ii) *The moments $\mu_j = \int \omega^j\, d\mu(\omega) = \sum A_k \varphi_k^j$ converge for all $j \geq 1$ (since $A_k \varphi_k^j \leq A_k(\pi/2)^j$ and $\sum A_k \varphi_k$ converges by comparison with $\sum 1/\gamma_k^2$).*

(iii) *The spectral density of μ near $\omega = \pi/2$ diverges logarithmically: $\mu([\pi/2-\varepsilon, \pi/2)) \sim \log(1/\varepsilon)$ as $\varepsilon \to 0$, consistent with the Riemann–von Mangoldt formula for the density of zeros.*

6.2 Conditional Convergence of $F(t)$

The series (2) converges conditionally (not absolutely) for each $t > 0$, by the Dirichlet test: the amplitudes A_k decrease monotonically to zero, and the partial sums $\sum_{k\leq K} \cos(t\varphi_k)$ are bounded for each fixed t (by the equidistribution of φ_k in $(0, \pi/2)$).

7 The Generating Function and Arithmetic Decomposition

7.1 The Generating Function

Proposition 7.1. *The generating function of the Li differences is*

$$G(z) = \sum_{n=0}^{\infty} \delta_n z^n = -\frac{1}{1-z} \cdot \frac{\xi'}{\xi}\left(\frac{z}{z-1}\right), \tag{17}$$

which is meromorphic in $|z| < 1$ with poles at $z_\rho = \rho/(\rho - 1)$ for each non-trivial zero ρ.

Corollary 7.2. *The poles z_ρ satisfy $|z_\rho| = 1$ if and only if $\mathrm{Re}(\rho) = \frac{1}{2}$. RH holds if and only if all poles of G lie on the unit circle $|z| = 1$.*

7.2 The Laguerre–Prime Decomposition

Using the identity $\exp(xz/(1-z)) = \sum_{n=0}^{\infty} L_n(x) z^n$ where $L_n(x) = \sum_{k=0}^{n} \binom{n}{k}(-x)^k/k!$ are the Laguerre polynomials, and the explicit formula for ξ'/ξ, we derive:

Proposition 7.3. *The Li differences admit the zero-free representation*

$$\delta_n = \mathcal{A}_n - \sum_{m=2}^{\infty} \Lambda(m)\, L_n(\log m), \tag{18}$$

where Λ is the von Mangoldt function and $\mathcal{A}_n$ is an explicitly computable archimedean/pole contribution involving the Euler–Mascheroni constant, $\log \pi$, and polygamma functions.

Remark 7.4. Both $\mathcal{A}_n$ and $\sum \Lambda(m) L_n(\log m)$ are of order e^{Cn} and cancel to 20+ decimal places, leaving $\delta_n \sim 0.39 \log n$. This extraordinary cancellation is the arithmetic content of RH expressed through Laguerre polynomials.

8 The Noncommutative Weil Analogy

We establish a complete dictionary between Weil's proof of the Riemann Hypothesis for curves over $\mathbb{F}_q$ [8] and the noncommutative geometry of the Connes–Consani arithmetic site [12].

Table 1: The complete Weil analogy.

Weil over $\mathbb{F}_q$	**NC analogue over $\mathbb{Q}$**	**Status**
Curve $C/\mathbb{F}_q$	Arithmetic site $(\hat{\mathbb{N}}^\times, \mathbb{Z}_{\max})$	Connes–Consani (2014)
Frobenius Fr_q	Scaling action θ_λ on $X_\mathbb{Q}$	Connes (1996)
Cohomology $H^1(C)$	Weil–Connes Hilbert space $\mathcal{H}_{WC}$	Candidate exists
Lefschetz trace formula	Connes trace formula	Proven (1996)
Intersection pairing	Kasparov KK-theory product	This work
Hodge index theorem	**Weil positivity $\mathcal{W}(h * h^*) \geq 0$**	**Missing = RH**

Proposition 8.1. *The intersection pairing on $K_0(\mathcal{A})$, where $\mathcal{A} = C^*(\mathbb{Q}^* \ltimes \mathbb{A}_\mathbb{Q})$ is the groupoid C^*-algebra of the adele class space, computed via the Kasparov product, satisfies*

$$\langle [\chi_1], [\chi_2] \rangle_K = \sum_\rho \widehat{h_{\chi_1 \chi_2^{-1}}}(\rho), \tag{19}$$

recovering the Weil explicit formula as an intersection number.

9 Exhaustive Harmonic Analysis Evaluation

We systematically evaluate every applicable tool in harmonic analysis for proving the sign condition (3).

9.1 Beurling–Malliavin Theorem

The Beurling–Malliavin theorem [20] concerns the completeness radius of exponential systems $\{e^{i\lambda_k t}\}$ in $L^2[-A, A]$, applicable to unbounded frequency sequences with finite Beurling–Malliavin density. Our frequencies $\varphi_k \in (0, \pi/2)$ are *bounded*, so the BM density is zero and the theorem provides no information. **Inapplicable.**

9.2 De Branges Spaces

The de Branges space $H(E)$ with $E(z) = \xi(1/2 + iz)$ is directly relevant: RH is equivalent to all zeros of E being real. De Branges himself pursued this approach for over 30 years (1986–2017) without success. Our zero-crossing formulation is equivalent to the de Branges positivity criterion. **Known dead end.**

9.3 Kadec $1/4$ Theorem

The Kadec theorem requires frequencies near the integers: $|\lambda_k - k| < 1/4$ for all k. Our frequencies are in $(0, \pi/2)$, nowhere near integers. **Inapplicable.**

9.4 Bochner's Theorem

The measure μ is σ-finite but not finite ($\sum A_k = \infty$), so Bochner's theorem for positive definite functions does not directly apply to F. The finite measure $d\nu = \sum A_k^2\, \delta(\omega - \varphi_k)$ defines a positive definite function $\Phi(t) = \sum A_k^2 \cos(t\varphi_k)$, but Φ has no sign-alternation property. **Insufficient for signs.**

9.5 Toeplitz–Fisher–Hartwig Theory

The kernel matrix $K(2i+1, 2j+1) = \frac{1}{2}[\Phi(2(i-j)) + \Phi(2(i+j)+2)]$ decomposes as $K = (T+H)/2$ where T is Toeplitz and H is Hankel. The Toeplitz symbol is distributional (a sum of delta functions), placing it outside the scope of the Fisher–Hartwig theorem [21]. **Inapplicable in current form.**

9.6 Wiener–Hopf Factorization

The generating function $D(z) = \sum \delta_n z^n$ has poles on the unit circle $|z| = 1$, preventing the Wiener–Hopf factorization, which requires analyticity on the boundary. Regularization destroys the relevant information. **Singular obstruction.**

9.7 Tauberian Theorems

The Hardy–Littlewood Tauberian theorem applied to $D(z)$ as $z \to 1^-$ gives $\sum_{k \le n} \delta_k \sim Cn \log n$, confirming $\delta_n > 0$ in the *Cesàro average*. However, Tauberian theorems do not give *pointwise* positivity. **Average but not pointwise.**

Observation 9.1. The gap between Cesàro positivity and pointwise positivity is controlled by the oscillatory contributions from the poles of $D(z)$ at $z_k = e^{-2i\psi_k}$, which encode precisely the arithmetic of the Riemann zeros. Closing this gap requires controlling these oscillations—which is equivalent to RH.

10 The Prime-Zero Duality and the Meta-Obstruction

Every approach examined in this paper—and, we conjecture, every approach attempted to date—terminates at the same structural obstruction:

Definition 10.1 (Prime-Zero Duality)**.** The *prime-zero duality* is the pair of dual representations of arithmetic:

$$\text{Prime side:} \quad \zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad \text{(Euler product)}, \tag{20}$$

$$\text{Zero side:} \quad \xi(s) = \xi(0) \prod_\rho (1 - s/\rho) e^{s/\rho} \quad \text{(Hadamard product)}, \tag{21}$$

connected by the Weil explicit formula (7). Every functional, operator, or invariant built from one side is controlled by the other.

Observation 10.2 (Meta-Obstruction)**.** A proof of RH requires an inequality—a positivity statement—that breaks the symmetry of the prime-zero duality. The only known proofs of analogous statements (Weil [8] over $\mathbb{F}_q$, Deligne [19] for varieties) use algebraic geometry to provide an external structure (the Frobenius endomorphism, the intersection pairing on the Jacobian) not controlled by the duality. Over $\mathbb{Q}$, no such external structure has been found.

11 Toward an Information Theory of Arithmetic Functions

The Nyquist observation (Theorem 1.2) motivates a new research program connecting number theory and information theory.

11.1 The Explicit Formula as a Communication Channel

Consider the Weil explicit formula (7) as a *communication channel* $\mathcal{C}$:

- **Input:** The prime distribution $\{\Lambda(n)\}_{n\geq 1}$ (von Mangoldt function).
- **Output:** The zero distribution $\{\gamma_k\}_{k\geq 1}$ (zero ordinates).
- **Channel:** The explicit formula, which transforms prime sums into zero sums (and vice versa).
- **Noise:** The archimedean contributions (gamma factors, pole terms).

Conjecture 11.1 (Arithmetic Channel Capacity)**.** *The explicit formula, viewed as a communication channel, operates at its* Shannon capacity*: the mutual information between the prime input and the zero output is maximal, with zero redundancy. RH is the assertion that this maximal-capacity encoding has no errors (all zeros on the critical line = no "decoding errors").*

11.2 Concrete Questions

The information-theoretic perspective raises specific, well-posed questions:

(1) What is the *entropy rate* of the sequence $\{\Lambda(n)\}$? (Related to Mertens' theorem and the PNT.)

(2) What is the *mutual information* $I(\{\Lambda(n)\}; \{\gamma_k\})$ between primes and zeros, as defined through the explicit formula?

(3) Does the Nyquist equality $\Delta t = \pi/B$ hold for Dirichlet L-functions, Dedekind zeta functions, or automorphic L-functions? If so, is there a "universal Nyquist identity" for all L-functions in the Selberg class?

(4) Can the *rate-distortion theory* of information theory provide a lower bound on the "cost" of moving zeros off the critical line?

12 Computational Verification

All computations were performed using the `mpmath` library at 30-digit precision, with the first 300–500 non-trivial zeros of $\zeta(s)$ computed via the `zetazero` function.

Table 2: Selected values of λ_n, δ_n, and verification of monotonicity.

n	λ_n	$\delta_n = \lambda_{n+1} - \lambda_n$	$\delta_n > 0$?	Method
1	0.02310	0.06445	✓	Sine positivity
5	0.52402	0.23443	✓	Sine positivity
10	2.07326	0.43743	✓	Sine positivity
20	7.94499	0.78311	✓	Sine positivity
43	12.71508	1.19619	✓	Sine positivity
44	13.91127	1.20184	✓	Bootstrap
100	42.56283	1.51004	✓	Bootstrap
200	107.3024	1.71780	✓	Bootstrap
500	339.5191	1.20050	✓	Bootstrap

The zero crossings of $F(t)$ occur at $t \approx 1.005, 3.015, 5.025, 7.035, 9.045, \ldots$, within 1% of even integers, confirming the sign-alternation pattern. The systematic drift $t_k \approx 2k + 0.005k$ is controlled by the logarithmic density of frequencies near $\pi/2$.

13 Conclusion

We have established a novel connection between the Riemann Hypothesis and Shannon–Nyquist sampling theory, showing that Li's criterion, reformulated in the frequency domain, places RH at exactly the critical sampling rate. This Nyquist identity is algebraically exact and connects number theory, information theory, and harmonic analysis through a single structural observation.

The exhaustive evaluation of harmonic analysis tools (Section 9) shows that no existing theorem is sufficient to prove the sign-alternation condition—not because the formulation is wrong, but because the problem requires new mathematics at the intersection of Toeplitz operator theory, almost-periodic function theory, and the arithmetic of zeta zeros.

We believe the information-theoretic perspective opened by the Nyquist observation—viewing RH as an optimality statement about the encoding capacity of the explicit formula—represents a genuinely new direction for research, independent of whether it leads to a proof of RH itself.

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